Rotating a Parabola

Saumya Singhal

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1 The Plan

First, we will start with the equation for a parabola in standard form, which is

$$Ax^2 + Bx + C = y.$$

We are then going to rotate the parabola by $l°$ clockwise. The goal is to get an equation in the "$y =$" format, because that is what my graphing calculator will accept.

1.1 How to rotate any equation

The method we are going to use for rotating our parabola is matrices transformations. In matrices transformations, we substitute the values

$$x \to ax + by$$

$$y \to -ay + bx$$

into the equation where

$$a = cos(l)$$

and

$$b = sin(l).$$

However, this transformation also flips the parabola upside down, so we must multiply our entire equation by -1 at the end.

1.2 Applying this to the Target

To begin the substitution process, we can write our target as the following equation:

$$A(ax + by)^2 + B(ax + by) + C = -ay + bx$$

2 Simplification

The above can be simplified as follows:

$$A(a^2x^2 + 2axby + b^2y^2) + Bax + Bby + C = -ay + bx$$
$$(Ab^2)y^2 + (2Aaxb + a + Bb)y + (Aa^2x^2 - bx + Bax + C) = 0$$

We now have an equation of a quadratic form, which can be solved for y using the quadratic formula. When the current values are substituted into the quadratic formula, it looks like

$$y = \frac{-(2Aaxb + a + Bb) \pm \sqrt{(2Aaxb + a + Bb)^2 - 4(Ab^2)(Aa^2x^2 - bx + Bax + C)}}{2Ab^2}$$

2.1 Isolating the Radical

To clear things up a bit, we isolate the radical portion of the equation, and begin to simplify it.

$$\sqrt{4A^2a^2b^2x^2 + a^2 + B^2b^2 + 4Aa^2bx + 4ABab^2x + 2Bab - (4Ab^2)(Aa^2x^2 - bx + Bax + C)}$$
$$\sqrt{4A^2a^2b^2x^2 + a^2 + B^2b^2 + 4Aa^2bx + 4ABab^2x + 2Bab - 4A^2a^2b^2x^2 + 4Ab^3x - 4ABab^2x - 4Ab^2C}$$

We can cancel out a few terms to obtain

$$\sqrt{\cancel{4A^2a^2b^2x^2} + a^2 + B^2b^2 + 4Aa^2bx + \cancel{4ABab^2x} + 2Bab - \cancel{4A^2a^2b^2x^2} + 4Ab^3x \cancel{-4ABab^2x} - 4Ab^2C}$$
$$= \sqrt{a^2 + B^2b^2 + 4Aa^2bx + 2Bab + 4Ab^3x - 4ACb^2}.$$

Distributing even further, we get

$$\sqrt{4Ab(a^2x + b^2x - bC) + a^2 + B^2b^2 + 2Bab}.$$

We can factor the last portion to further simplify to

$$\sqrt{4Ab(a^2x + b^2x - bC) + (a + Bb)^2}.$$

However, we must remember that

$$a = \sin(l)$$
$$b = \cos(l)$$

and

$$\sin^2(l) + \cos^2(l) = 1.$$

Therefore,

$$a^2 + b^2 = 1$$

and

$$a^2x + b^2x = x$$

Therefore, our radical becomes

$$\sqrt{4Ab(x - bC) + (a + Bb)^2}.$$

2.2 Substituting the Radical

Now that we have simplified the radical completely, we are ready to pop it back into the equation

$$y = \frac{-(2Aaxb + a + Bb) \pm \sqrt{4Ab(x - bC) + (a + Bb)^2}}{2Ab^2}$$

3 The Final Equation

Finally, after multiplying by -1 we obtain the equation

$$y = \frac{-(2Aabx + a + Bb) \pm \sqrt{4Ab(x - bC) + (a + Bb)^2}}{-2Ab^2}.$$

4 Graphing

This is the graph of $y = x^2$ rotated by $150°$.